

DEDUCTION OF THE QUANTUM NUMBERS OF LOW-LYING STATES
OF THE $(e^+e^+e^-e^-)$ SYSTEM FROM SYMMETRY CONSIDERATION

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ABSTRACTS The features of the low-lying spectrum and a complete set of quantum numbers for the $(e^+e^+e^-e^-)$ system have been deduced based on symmetry. The existence of a low odd-parity $L=1$ excited state with the two e^- coupled to $s_1 = 0$, and two e^+ coupled to $s_2 = 1$ (or $s_1 = 1$ and $s_2 = 0$) and a low even-parity $L=0$ excited state with $s_1 = s_2 = 1$ have been anticipated. A 2-dimensional $(e^+e^+e^-e^-)$ system has also been discussed.

PACS: 33.20.Sn, 31.90.+s, 03.65.-w

KEYWORDS: $(e^+e^+e^-e^-)$ system, symmetry consideration, classification of states, energy spectrum.

The systems composed of a few electrons and positrons are known to exist in nature. The existence of the (e^+e^-) with infinite bound states is obvious. The $(e^+e^-e^-)$ or $(e^+e^+e^-)$ has been investigated by many authors. In 1977 Hill proved that these two 3-body systems are bound [1]. The ground states and a number of resonances have been identified ([2,3] and references there in). For the $(e^+e^+e^-e^-)$ a direct experimental observation has not yet been reported. However, a few theoretical calculations [4-8] indicate that this system is bound. A recent (and more accurate) calculation reports that the ground state has an energy $E_0 = -0.515989$ (atomic units are used in this paper) and a radius $r_0 = 3.608$ [8]. From r_0 one can estimate the moment of inertia I_0 of the ground state. It was found that the $\frac{\hbar^2}{2I_0}$ is only about $\frac{1}{50} |E_0|$. Since for Coulombic systems the excited states would be in general much larger than the ground state in size (as shown in [6]), it implies that an excitation of collective rotation is very easy to realize in an excited state due to having a large moment of inertia. . Therefore, besides the orbital angular momentum $L=0$ resonances predicted before [5,6], the existence of a number of resonances with $L \neq 0$ is very possible. Thus, the spectrum of the $(e^+e^+e^-e^-)$ would be rich. However, a precise calculation is difficult. Before solving the Schrödinger equation, if the qualitative feature can be more or less understood, it would be very helpful to understand the coming results from experiments and theoretical calculations. It is believed that symmetry will play an essential role in the spectrum. Based on the permutation symmetry, Kinghorn and Poshusta have proposed a primary scheme for the classification of states [6]. On the other hand, the feature of a quantum state depends on how the wave function is distributed in the coordinate space. It turns out that the distribution is decisively affected by symmetry (not only by the permutation symmetry, as we shall see). In this paper the effect of symmetry has been investigated in detail, thereby the geometric feature of low-lying states has been extracted, a complete set of quantum numbers (including those proposed in [6]) has been introduced for the classification.

Let H be the internal Hamiltonian for the $(e^+e^+e^-e^-)$. Let particles 1 and 2 be electrons, 3 and 4 be positrons. It was pointed out by Kinghorn and Poshusta that H is invariant under the permutations

$((e)), ((12)), ((34)), ((12))((34)), p(1324), p(1423), ((13))((24)),$ and $((14))((23)),$

here the $((12))$ denotes an interchange of 1 and 2, $p(1324)$ denotes a cyclic permutation, etc.. These permutations form a subgroup of the S_4 group isomorphic to the point group D_{2d} [6]. Therefore the eigenstates of H can be classified according to the representations μ of the D_{2d} group, $\mu = A_1, A_2, B_1, B_2,$ or E . These representations can be induced by the following idempotents:

$$e^{A_1} = \frac{1}{8}[1 + ((13))((24))][1 + ((12))][1 + ((34))] \quad (1,1)$$

$$e^{A_2} = \frac{1}{8}[1 - ((13))((24))][1 - ((12))][1 - ((34))] \quad (1,2)$$

$$e^{B_1} = \frac{1}{8}[1 + ((13))((24))][1 - ((12))][1 - ((34))] \quad (1,3)$$

$$e^{B_2} = \frac{1}{8}[1 - ((13))((24))][1 + ((12))][1 + ((34))] \quad (1,4)$$

$$e^{E_{11}} = \frac{1}{4}[1 - ((12))((34)) + ((13))((24)) - ((14))((23))] \quad (1,5)$$

$$e^{E_{22}} = \frac{1}{4}[1 - ((12))((34)) - ((13))((24)) + ((14))((23))] \quad (1,6)$$

Evidently, the sum of these six idempotents is equal to one.

On the other hand, H is also invariant under rotation and space inversion. Besides, the total spin s_1 of particles 1 and 2 and the total spin s_2 of 3 and 4 are also good quantum numbers. Let the spatial wavefunctions of the eigenstates be denoted as Ψ . Evidently, Ψ should be further classified according to the total orbital angular momentum L , parity Π , s_1 and s_2 . In particular, we have

$$((12))\Psi = (-1)^{s_1}\Psi \quad (2,1)$$

and

$$((34))\Psi = (-1)^{s_2}\Psi \quad (2,2)$$

Equations (2,1) and (2,2) are additional conditions imposed on the eigenstates of H .

For an eigenstate Ψ belonging to the A_1 -representation, it is evident from eq.(1,1) that

$$((12))\Psi = \Psi, ((34))\Psi = \Psi, \text{ and } ((13))((24))\Psi = \Psi$$

Therefore this state has $s_1=0$, $s_2=0$, and $\Lambda=1$, where Λ is the eigenvalue of $((13))((24))$. Similarly, the states belonging to the A_2 -representation have $s_1=1$, $s_2=1$, and $\Lambda = -1$; the states of B_1 have $s_1=1$, $s_2=1$, and $\Lambda=1$; and the states of B_2 have $s_1=0$, $s_2=0$, and $\Lambda=-1$ (cf. Table 1).

In the case of the E -representation, there are two degenerate states for an eigenenergy. They can be denoted as $e^{E_{11}}\Psi$ and $((12))e^{E_{11}}\Psi$ (or $e^{E_{22}}\Psi$ and $((12))e^{E_{22}}\Psi$). However, neither the $e^{E_{11}}\Psi$ nor the $((12))e^{E_{11}}\Psi$ fulfills (2.1) and (2.2), therefore each of them is not a physical solution. Nonetheless, the two linear combinations

$$\Psi_a = [1 - ((12))]e^{E_{11}}\Psi \quad (3,1)$$

and

$$\Psi_b = [1 + ((12))]e^{E_{11}}\Psi \quad (3,2)$$

are physical solutions. One can prove that $((12))\Psi_a = -\Psi_a$, $((34))\Psi_a = \Psi_a$, and $((13))((24))\Psi_a = \Psi_b$. Therefore the Ψ_a has $s_1=1$ and $s_2=0$, but it does not have the quantum number Λ . For convenience, this state is denoted as $\Lambda=0$. Similarly, the Ψ_b has $s_1=0$, $s_2=1$, and $\Lambda=0$. The quantum numbers L, Π , and μ are sufficient to label the symmetry of a series of states. Thus a state is labeled as $L_i^\Pi\{\mu\}$ in the following, where i denotes the ordering of the state in the $(L\Pi\mu)$ series. More explicitly, the spatial wavefunction of a $L_i^\Pi\{\mu\}$ state is denoted in more detail as $F_{M,s_1s_2}^{L\Pi\Lambda}(1234)$, M is the Z -component of L . The relation between μ and the set $(s_1s_2\Lambda)$ is listed in Table 1.

In order to facilitate the analysis, a body frame $i'-j'-k'$ is introduced. Then the wave function can be expanded as

$$F_{M,s_1s_2}^{L\Pi\Lambda}(1234) = \sum_Q D_{QM}^L(-\gamma, -\beta, -\alpha) F_{Q,s_1s_2}^{L\Pi\Lambda}(1'2'3'4') \quad (4)$$

where Q is the component of L along the third body axis k' , $\alpha\beta\gamma$ are the Euler angles, D_{QM}^L is the Wigner function of rotation. Here the (1234) and $(1'2'3'4')$ specify that the coordinates are relative to a fixed frame and to the body-frame, respectively. $F_{Q,s_1s_2}^{L\Pi\Lambda}(1'2'3'4')$ is called a Q -component. We shall show that the Q -components are seriously constrained by symmetry.

Evidently, the distribution of the wave function in the coordinate space depends first on the interactions. For the $(e^+e^+e^-e^-)$ system, if the particles form a shape where the repulsive force between the pair of electrons (or positrons) can not be over compensated by attractive force, then the shape is not important for low-lying states, and the amplitude of wave function is expected to be small at this shape. This is the case, e.g., that the particles form a regular tetrahedron or an quadrilateral with the pair of electrons (positrons) located at adjacent vertexes. On the other hand, if the particles form an quadrilateral with the pair of electrons (positrons) located at the two ends of a diagonal, then the configuration is favorable to binding, where each repulsive bond can be over compensated. When the particles form a square with the same kind of particles located at the two ends of each diagonal, this particular configuration is labeled as SQ hereafter.

Let the third body axis k' be normal to the plane of the SQ, we have

(i) a rotation of the SQ about k' by 180° is equivalent to a space inversion, thus we have

$$(-1)^Q F_{Q,s_1s_2}^{L\Pi\Lambda}(SQ) = \Pi F_{Q,s_1s_2}^{L\Pi\Lambda}(SQ) \quad (5)$$

here the (SQ) implies that the coordinates form a SQ.

(ii) a rotation about k' by 180° is equivalent to $((12))((34))$, thus we have

$$(-1)^Q F_{Q,s_1s_2}^{L\Pi\Lambda}(SQ) = (-1)^{s_1+s_2} F_{Q,s_1s_2}^{L\Pi\Lambda}(SQ) \quad (6)$$

(iii) Let i' be parallel to \vec{r}_{12} lying along a diagonal, a rotation about i' by 180° is equivalent to $((34))$, thus we have

$$(-1)^L F_{Q,s_1s_2}^{L\Pi\Lambda}(SQ) = (-1)^{s_2} F_{Q,s_1s_2}^{L\Pi\Lambda}(SQ) \quad (7)$$

(iv) a rotation about k' by -90° is equivalent to $p(1324)$, thus we have

$$i^Q F_{Q,s_1s_2}^{L\Pi\Lambda}(SQ) = \Lambda(-1)^{s_2} F_{Q,s_1s_2}^{L\Pi\Lambda}(SQ) \quad (\text{if } \Lambda = \pm 1) \quad (8.1)$$

$$i^Q F_{Q,s_1s_2}^{L\Pi\Lambda}(SQ) = (-1)^{s_2} F_{Q,s_2s_1}^{L\Pi\Lambda}(SQ) \quad (\text{if } \Lambda = 0) \quad (8.2)$$

These equations impose a very strong constraint on $F_{Q,s_1s_2}^{L\Pi\Lambda}(SQ)$. If they can not be fulfilled, $F_{Q,s_1s_2}^{L\Pi\Lambda}$ must be zero at the SQ. In this case, a nodal surface exists at the SQ to prevent the Q-component $F_{Q,s_1s_2}^{L\Pi\Lambda}$ from accessing the SQ. Evidently, this nodal surface arises purely from symmetry, it is called an inherent nodal surface [9,10]. The existence and the locations of the inherent nodal surfaces do not at all depend on dynamics, it depends only on the transformation property of the wavefunction under symmetric operations (rotation, space inversion, and particle permutation), this property is determined by the quantum numbers L, Π , and μ . All the Q-components of the $L \leq 2$ states allowed to access the SQ are listed in Tab.1, they are called SQ-accessible components.

All the other regular shapes in the neighborhood of the SQ have a weaker geometric symmetry. For example, when a diagonal of a SQ becomes longer or shorter, the shape becomes to a diamond. At this shape the equations (5) to (7) hold, but (8) does not hold. Therefore the wavefunction is less constrained at the diamonds. Hence, once a wavefunction can access the SQ, it can access a broad domain surrounding the SQ. Therefore the SQ-accessible components are inherent nodeless in this broad domain, where the distribution of wave function can be optimized to favor binding.

When the particles form a straight chain with each adjacent pair separated equidistantly and with each adjacent pair containing opposite charges, the configuration is denoted by CH. Evidently, at the CH, the repulsion can also be over compensated by attraction. However, all the four attractive bonds at the SQ act effectively while only three bonds at the CH do. Thus, the CH is less favorable than the SQ, and the domain surrounding the CH is the second important domain for the low-lying states.

Let \mathbf{k}' be normal to and \mathbf{i}' be parallel to the straight chain, let the ordering of the particles along the chain is 1423. Then, at the CH, a rotation about \mathbf{k}' by 180° is equivalent to a space inversion and also equivalent to $p_{13}p_{24}$, a rotation about \mathbf{i}' by an arbitrary angle δ causes no changes. Thus we have

$$(-1)^Q F_{Q,s_1 s_2}^{L\Pi\Lambda}(CH) = \Pi F_{Q,s_1 s_2}^{L\Pi\Lambda}(CH) \quad (9)$$

$$(-1)^Q F_{Q,s_1 s_2}^{L\Pi\Lambda}(CH) = \Lambda F_{Q,s_1 s_2}^{L\Pi\Lambda}(CH) \quad (\text{if } \Lambda = \pm 1) \quad (10.1)$$

$$(-1)^Q F_{Q,s_1 s_2}^{L\Pi\Lambda}(CH) = F_{Q,s_2 s_1}^{L\Pi\Lambda}(CH) \quad (\text{if } \Lambda = 0) \quad (10.2)$$

and

$$(1 - i\delta L_{i'} + \frac{1}{2}(i\delta L_{i'})^2 + \dots) F_{Q,s_1 s_2}^{L\Pi\Lambda}(CH) = F_{Q,s_1 s_2}^{L\Pi\Lambda}(CH) \quad (11)$$

where $L_{i'}$ is the projection of L along \mathbf{i}' . Evidently, (11) is fulfilled if

$$L_{i'} F_{Q,s_1 s_2}^{L\Pi\Lambda}(CH) = 0 \quad (\text{for all } Q) \quad (12)$$

(12) can be rewritten as

$$\begin{aligned} & \sqrt{(L-Q)(L+Q+1)} F_{Q+1,s_1 s_2}^{L\Pi\Lambda}(CH) + \sqrt{(L+Q)(L-Q+1)} F_{Q-1,s_1 s_2}^{L\Pi\Lambda}(CH) \\ & = 0 \quad (\text{for all } Q) \end{aligned} \quad (13)$$

From this set of homogeneous linear equation, the $F_{Q,s_1 s_2}^{L\Pi\Lambda}(CH)$ are not zero only if

$$(-1)^Q = (-1)^L \quad (14)$$

From (9), (10), and (14) the CH-accessible components can be identified as listed in Tab.1. Incidentally, for a state with $L \geq 2$, the components with different $|Q|$ will be mixed up according to (13).

Let the $L_1^\Pi\{\mu\}$ state, the lowest one of the $(L\Pi\mu)$ series, be called a first-state (the subscript i will be omitted if $i = 1$). It is well known that higher states in

general contain more nodal surfaces. Thus it is reasonable to assume that the first-states would prefer to contain nodal surfaces as least as possible. Therefore they would be dominated by the component(s) containing least inherent nodal surfaces. In this paper the analysis is based on the inherent nodal structure in the two most important domain, i.e., the domain surrounding the SQ and the domain surrounding the CH. There is no a sharp border to separate these domains. Owing to the long range character of the Coulomb force, the distribution of the wave functions is anticipated to be broad, and the wave functions will extend from one domain to another. Hence, the component which is inherent nodeless in both the SQ and CH domains is superior. In Table 1 there are four such superior components denoted as SQ,CH. Accordingly, there are four states with $L \leq 2$ superior in binding. Each of them is dominated by a superior component., where the wave function is anticipated to be mainly distributed in the SQ-domain but extend to the CH-domain without nodal surfaces. These states are called the SQ1-states as listed in Table 2.

There is a type of component being SQ-accessible but CH-inaccessible (denoted as SQ in Table 1). For the $1^+\{A_2\}, 2^+\{B_2\}, 2^-\{E\}$, this type of component is much better than the other components of them. Thus these three first-states would be dominated by this type of component, and they are called the SQ2-states as listed in Table 2. The internal energy of a SQ2-first-state is higher than a SQ1-first-state because the former contains an inherent nodal surface while the latter does not. .

There is a type of component being SQ-inaccessible but CH-accessible. They can be further classified into three kinds. The first kind is not only CH-accessible but also rectangle-accessible (denoted by CH1 in Table1). The states dominated by this kind of components are the $0^+\{B_1\}$ and $2^+\{B_1\}$, they are called the CH1-states as listed in Table 2. The second kind denoted as CH2 in Table 1 is rectangle-accessible if and only if the particles at the two ends of a diagonal have opposite charges (otherwise, it is rectangle-inaccessible). The states dominated by this kind of components are the $1^-\{A_2\}$ and $1^-\{B_2\}$, they are called the CH2-states. The third kind denoted as CH3 in Table 1 is rectangle-inaccessible (disregarding how the particles are located at the vertexes). The states dominated by this kind of components are the $0^+\{E\}$ and $2^+\{E\}$, they are called the CH3-states. The wavefunctions of the CH1-, CH2-, and CH3-first-states would be mainly distributed around a straight chain. Among them the internal energy of the CH1-first-states would be lower due to being rectangle-accessible, the CH3-first states would be higher due to being rectangle-inaccessible (thus more nodal surfaces are contained).

There is a type of components being both SQ- and CH-inaccessible but diamond-accessible (denoted as D-S-D in Table 1). The $0^+\{B_2\}, 1^+\{B_1\}$, and $2^+\{A_2\}$ would be dominated by this type of component because they do not have a better choice. In these states coplanar structure (e.g., a diamond) would exist. However, the inherent nodal surface at the SQ will induce a coplanar oscillation back and forth around the SQ, therefore the energy of these states are high. They are called the D-S-D states listed in Table 2

There is a type of components being not allowed to access any coplanar struc-

ture, they also can not access the CH, but they can access a regular tetrahedron (denoted as T-S-T in Table 1). The $0^-\{B_1\}$, $1^+\{E\}$, \dots would be dominated by this type of component, where the wavefunction is mainly distributed around a tetrahedron. However, the inherent nodal surface at the coplanar structure will induce an oscillation so that the tetrahedron would be transformed to another tetrahedron via a coplanar structure. The energy of these states are also high. They are called the T-S-T states listed in Table 2.

There is also a type of components being not allowed to access all the above mentioned shapes, namely the SQ, rectangle, diamond, CH, and the regular tetrahedrons (denoted as a blank block in Table 1). This type is very unfavorable in binding due to containing many nodal surfaces at the regular shapes. However, the $0^+\{A_2\}$, $0^-\{A_1\}$, \dots states contain only this type of component. The energy of these states must be very high. They are listed at the bottom of Table 2.

A complete classification of the first-states according to the inherent nodal structure of their dominant component is given in Table 2. The $L \geq 3$ first-states can be classified as well. It is noted that in Table 2 the states in an upper row contains fewer inherent nodal surface; the upper, the fewer. For the first-states with a given L , the ordering of their locations in Table 2 (in descending order) is anticipated to be the ordering of their energies (in ascending order): the SQ1-states at the top are the lowest which are essentially inherent nodeless, while the states at the bottom are the highest which contain a number of inherent nodal surfaces at regular shapes. Nonetheless, the SQ2- and CH1-states both contain essentially one inherent nodal surface, therefore their ordering can not be simply determined by symmetry. This is also the situation of the D-S-D and T-S-T states. In the second-states and even higher states, in addition to inherent nodal surfaces, more nodal surfaces arising from a pure dynamical reason (not from symmetry) will be contained. We are not going to the details of these states. Nonetheless, they will also be seriously affected by the inherent nodal structure as listed in Table 1.

Incidentally, the collective rotation energy of a coplanar structure depends on the orientation of the structure relative to L . When the third body axis k' is normal to the plane of the coplanar structure, the plane would be better normal to L in a Q-component with a larger $|Q|$. Thus, for a coplanar shape, the larger the $|Q|$, the larger the moment of inertia, and thereby the smaller the rotation energy. For example, although both the $2^+\{B_1\}$ and $2^+\{A_1\}$ are SQ1-states with $L=2$, the collective rotation energy of the former is smaller because the SQ structure would appear in $|Q|=2$ component (refer to Table 1). This effect would affect the ordering of levels. How serious is the effect remains to be clarified.

It is anticipated that the low-lying spectrum would be dominated by the SQ1-, SQ2-, and CH1-states. In particular, the ground state would be the $0^+\{A_1\}$. The candidate of the first excited state would be the odd-parity $1^-\{E\}$ and the $0^+\{B_1\}$. The $1^-\{E\}$ is two-fold degenerate because the (s_1s_2) has two choices (0,1) and (1,0). The internal energy of the $0^+\{B_1\}$ as a CH1-state is higher than the $1^-\{E\}$ as a SQ1-state, however the $0^+\{B_1\}$ does not contain collective rotation energy, thus a compensation can be gained. It is a pity that experimental and theoretical data

are so few that most of the predictions can not be checked at present. Nonetheless, owing to the work by Kinghorn and Poshusta [6], the predictions on $L=0$ even-parity states can be partially checked. The results from [6] are listed in Table 3 to be compared with Table 2 and Table 1. In Table 3 the $0^+\{A_1\}$ is the lowest just as anticipated. Besides, the ordering of levels is just as anticipated; i.e., the energies of the SQ1-state, CH1-state, CH3-state, D-S-D state, and the state belonging to the last row of Table 2 are in ascending order. Thus, in general, the spectrum from [6] supports the above analysis and the classification.

In the above analysis the features of the wavefunctions has been demonstrated. Obviously, all these features deduced from symmetry consideration together with the suggested classification scheme need a further check. In particular, although the analysis covers all the $L \leq 2$ states, it does not mean that all of them can be experimentally observed. The observation depends on the width of the resonance, which is a topic not yet touched in this paper.

The above discussion can be easily generalized to a 2-dimensional ($e^+e^+e^-e^-$) system. In this case $\Pi = (-1)^L$ holds always, therefore the label $L_i\{\mu\}$ is sufficient to denote a state. The accessibility of the regular shapes to the $L_i\{\mu\}$ states is summarized in Table 4, which leads to a classification of states as listed in Table 5. The ground state is also a $0\{A_1\}$ state. There are three lower excited states, namely the $1\{E\}$, $0\{A_2\}$ and $0\{B_1\}$. It is noticeable that the $0\{A_2\}$ is now low. This point is very different from the 3-dimensional system where the $0^+\{A_2\}$ is very high. We can further predict that the $2\{B_1\}$ is the lowest $L=2$ state.

In the case of a 2-dimensional biexcitons in a quantum well [11], the effective mass of the hole is not necessary equal to the mass of the particle, therefore the Hamiltonian is no longer invariant to the D_{2d} group. In this case, the above discussion should be modified.

The procedure of analysis proposed in this paper is quite general for investigating the qualitative feature of low-lying states. In fact it can be generalized to investigate different kinds of systems with different numbers of particles [12-13]. In any case, one has to clarify which domains in the coordinate space are important (it depends on the interactions) to the low-lying states, and to clarify the inherent nodal structure (it depends purely on symmetry) in these important domains. The inherent nodal structure would provide a solid frame to classify the states. Based on the inherent nodal structure, one can understand not only the behavior of an individual system, but also the systematics of spectra and the similarity of different systems.

ACKNOWLEDGMENT This work is supported by the NNSFC of the PRC, and by a fund from the National Educational Committee of the PRC.

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Π	s_1s_2	Λ	μ	Q=0	Q=0	$ Q =1$	Q=0	$ Q =1$	$ Q =2$
+	00	1	A_1	SQ, CH1			SQ, CH1		CH, D-S-D
+	00	-1	B_2	D-S-D			D-S-D		SQ
+	11	1	B_1	CH1	D-S-D		CH1		SQ, CH1
+	11	-1	A_2		SQ				D-S-D
+	01, 10	0	E	CH3		T-S-T	CH3	T-S-T	CH2
-	00	1	A_1						T-S-T
-	00	-1	B_2		T-S-T	CH2			
-	11	1	B_1	T-S-T			T-S-T		
-	11	-1	A_2			CH2			T-S-T
-	01, 10	0	E			SQ, CH1		SQ	

Table 1, The accessibility of the regular shapes to the Q-components of the $L^\Pi\{\mu\}$ states. A block with a SQ (CH) denotes that the corresponding component can access the SQ (CH). If both the SQ and CH are contained in a block, both can be accessed.

SQ1-states	$0^+\{A_1\}, 1^-\{E\}, 2^+\{B_1\}, 2^+\{A_1\}$
SQ2-states	$1^+\{A_2\}, 2^+\{B_2\}, 2^-\{E\}$
CH1-states	$0^+\{B_1\}, 2^+\{B_1\}$
CH2-states	$1^-\{B_2\}, 1^-\{A_2\}$
CH3-states	$0^+\{E\}, 2^+\{E\}$
D-S-D states	$0^+\{B_2\}, 1^+\{B_1\}, 2^+\{A_2\}$
T-S-T states	$0^-\{B_1\}, 1^+\{E\}, 2^-\{A_1\}, 2^-\{B_1\}, 2^-\{A_2\},$ $0^+\{A_2\}, 0^-\{A_1\}, 0^-\{B_2\}, 0^-\{A_2\}, 0^-\{E\}, 1^+\{A_1\}, 1^+\{B_2\}, 1^-\{A_1\}, 1^-\{B_1\}, \dots$

Table 2, A classification of the $L \leq 2$ states.

	type	energy
$0^+\{A_1\}$	SQ1	-0.5160
$0^+\{B_1\}$	CH1	-0.4994
$0^+\{E\}$	CH3	-0.3300
$0^+\{B_2\}$	D-S-D	-0.3145
$0^+\{A_2\}$		-0.3121

Table 3, The energies of the 0^+ first-states from [6] in atomic unit.

	L=0	L=1	L=2
A ₁	SQ, CH1		CH1, D-S-D
B ₂	D-S-D	CH2	SQ
B ₁	CH1, D-S-D		SQ, CH1
A ₂	SQ	CH2	D-S-D
E	CH2	SQ, CH1	CH2

Table 4, The accessibility of regular shapes to the eigenstates of a 2-dimensional ($e^+e^+e^-e^-$) system. Refer to Table 1.

SQ1-states	$0\{A_1\}, 1\{E\}, 2\{B_1\}$
SQ2-states	$0\{A_2\}, 2\{B_2\}$
CH1-states	$0\{B_1\}, 2\{A_1\}$
CH2-states	$0\{E\}, 1\{B_2\}, 1\{A_2\}, 2\{E\}$
D-S-D states	$0\{B_2\}, 2\{A_2\}$
	$1\{A_1\}, 1\{B_1\}$

Table 5, A classification of 2-dimensional $L \leq 2$ states. Refer to Table 2.